

# Covariant Hysteretic Constitutive Theory for Maxwell's equations: Application to Axially Rotating Media

**Alison C Hale and Robin W Tucker**

Department of Physics, Lancaster University, Lancaster, LA1 4YB, UK

and

Cockcroft Institute of Accelerator Science and Technology, Daresbury, Keckwick Lane, Daresbury, WA4 4AD, UK

E-mail: [a.c.hale@lancaster.ac.uk](mailto:a.c.hale@lancaster.ac.uk), [r.tucker@lancaster.ac.uk](mailto:r.tucker@lancaster.ac.uk)

**Abstract.** This paper explores a class of non-linear constitutive relations for materials with memory in the framework of covariant macroscopic Maxwell theory. Based on earlier models for the response of hysteretic ferromagnetic materials to prescribed slowly varying magnetic background fields, generalized models are explored that are applicable to accelerating hysteretic magneto-electric substances coupled self-consistently to Maxwell fields. Using a parameterized model consistent with experimental data for a particular material that exhibits purely ferroelectric hysteresis when at rest in a slowly varying electric field, a constitutive model is constructed that permits a numerical analysis of its response to a driven harmonic electromagnetic field in a rectangular cavity. This response is then contrasted with its predicted response when set in uniform rotary motion in the cavity.

## 1. Introduction

In recent years developments in technology have enabled the synthesis of new materials with interesting mechanical and electromagnetic properties. These have, in turn, led to new advances in technology as well as challenges in our understanding of physics at the interface between classical and quantum behaviours. In particular many meta-materials on the mesoscopic scale have a rich electromagnetic phenomenology. In the presence of high frequency or high intensity electromagnetic fields many exhibit local or non-local non-linear electromagnetic constitutive properties. At the other extreme some materials exhibit a delayed response to slowly varying electric or magnetic fields. *Ferromagnetic* media respond with a delayed magnetisation while *ferroelectrics* respond with a delayed electrical polarisation. They also exhibit an ability to maintain a saturated internal magnetisation (ferromagnets) or internal electric polarisation (electrets). Anisotropic *magneto-electric* materials also exist that sustain both types of fields. When the internally induced fields are not uniquely determined by any externally applied field one often says that the material exhibits electromagnetic constitutive properties with memory. When the external fields vary periodically with time and the induced fields respond periodically, the process is often referred to as hysteretic and the corresponding non-linear constitutive relation between them may exhibit a discontinuous or branched structure to account for this [1], [2], [3]. However such terminology is often restricted to processes where the induced fields in the medium saturate at some level and where the *shape* of the resulting hysteresis loop, obtained by displaying the magnitude of the drive field against the magnitude of the induced magnetisation or polarisation is independent of time. When the drive field is aperiodic in time the memory effects may exhibit a more complex Lissajoux structure in the hysteretic response, particularly if the time dependent drive field contains more than one dominant harmonic component.

A rapidly varying time-harmonic electromagnetic drive field may also induce both electric and magnetic polarisations with magnitudes dependent on the magneto-electric susceptibility 3-tensors of the medium. For materials with memory such susceptibilities will depend non-linearly on the electromagnetic field in the medium and the resulting hysteretic response will involve both induced polarisations. In addition to hysteretic responses all materials exhibit spatial and temporal dispersion to some degree and may also sustain induced electric currents as a result of their conductivity. Even in homogeneous and isotropic media the detailed description of such media in terms of their basic constituents and micro-structure is rarely possible and recourse to a parametrised model becomes necessary [4]. The parameters of such a phenomenological model are sought from experiment over some range and the model tentatively extrapolated outside that range. The degree of extrapolation is often dictated by comparing the model with experiment.

In this article a model is constructed that can describe a rigid non-dispersive, rapidly, uniformly rotating, hysteretic medium in an external time-harmonic electromagnetic field given its behaviour at rest. The model assigned to the medium at rest is motivated by a non-covariant model constructed by Coleman and Hodgdon [5], [6]. By exploiting the inherent spacetime covariance of the macroscopic Maxwell equations such a model can be embedded in a covariant formulation and coupled naturally to such equations. This system can then be reduced to a coupled differential system in terms of electromagnetic fields and spatial tensors describing the magnetisation and polarisation in an arbitrarily moving medium. To illustrate

how such a system can have practical implications these equations are attacked numerically for an axially rotating hysteretic ferroelectric in a perfectly conducting cavity containing electromagnetic fields driven by an external harmonic electric current.

## 2. Macroscopic Covariant Electrodynamics

A theory will be said to admit a spacetime covariant formulation if it can be expressed in terms of tensor field equations on spacetime. The covariant theory of macroscopic electrodynamics benefits from a formulation in terms of differential forms. Aside from its elegance it makes precise a number of conceptual terms used in the interpretation of the theory of accelerated media and provides a suite of economical tools that streamline calculations. Such tools include the exterior product, exterior derivative, Lie derivative, covariant derivative, interior derivative and Hodge map [7]. These operations find their natural setting on arbitrary manifolds of arbitrary dimension. In this article they are employed on 3-dimensional Euclidean space and 4-dimensional Minkowski spacetime. In the former case the Hodge map is denoted by  $\#$  and satisfies the rule  $\#\# = 1$  when acting on all (time-dependent) differential forms on space. The exterior derivative on such forms is denoted  $\underline{d}$  and satisfies  $\underline{d}\underline{d} = 0$ . These two identities suffice to determine the many interrelations between the curl operator ( $\#\underline{d}$ ) and div operator ( $\#\underline{d}\#$ ) in Euclidean space. In spacetime the Hodge map is denoted by  $\star$  and satisfies  $\star\star = (-1)^{1+p}$  when acting on  $p$ -forms on spacetime. The exterior derivative of such  $p$ -forms on spacetime is denoted  $d$  and satisfies  $dd = 0$ . Although gravitation is negligible in the following the relations between the frame-dependent spatial description of electromagnetism and its frame-independent spacetime description and the Hodge map that enters via the constitutive modelling require a metric tensor field  $\mathbf{g}$  for their formulation. To this end one introduces a set of independent cobasis 1-forms  $e^0, e^1, e^2, e^3$  on spacetime and writes the Minkowski metric tensor field

$$\mathbf{g} = -e^0 \otimes e^0 + \underline{\mathbf{g}} \quad (1)$$

where  $\underline{\mathbf{g}} = \sum_{k=1}^3 e^k \otimes e^k$  is the induced metric tensor on Euclidean space. In terms of these forms,  $\star 1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3$ , and  $\#1 = e^1 \wedge e^2 \wedge e^3$ . The macroscopic Maxwell system on spacetime is defined in terms of the electromagnetic 2-form  $F$ , a polarisation 2-form  $\Pi$  and a 4-current 3-form  $\mathcal{J}$ :

$$dF = 0 \quad (2)$$

$$d\star G = \mathcal{J} \quad (3)$$

where

$$G = \epsilon_0 F + \Pi \quad (4)$$

in terms of the permittivity of free space  $\epsilon_0$ . It is the responsibility of constitutive theory to provide auxiliary equations to render an augmented system deterministic. Thus constitutive auxiliary conditions should specify the dependence of  $\Pi$  and the 4-current  $\mathcal{J}$  (subject to 4-current conservation,  $d\mathcal{J} = 0$ ), on  $F$  and its possible derivatives. Such conditions may be local or non-local, linear or non-linear, algebraic or differential.

An arbitrary observer in spacetime can be associated with the integral curve of an arbitrary future pointing unit time-like vector field  $V$ :  $\mathbf{g}(V, V) = -1$ . Given  $\mathbf{g}$ , such a vector field determines the 1-form  $\tilde{V}$  by

$$\tilde{V} = \mathbf{g}(V, -) \quad (5)$$

It is convenient to refer to such a  $V$  as a frame (in spacetime) since it determines the components of spacetime tensors measured by the associated observer. Thus the 2-forms  $F, G, \Pi$  admit the orthogonal decompositions with respect to  $V$ :

$$F = e^V \wedge \tilde{V} - cB^V = e^V \wedge \tilde{V} + \star \left( cb^V \wedge \tilde{V} \right) \quad (6)$$

$$G = d^V \wedge \tilde{V} - \frac{H^V}{c} = d^V \wedge \tilde{V} + \star \left( \frac{h^V}{c} \wedge \tilde{V} \right) \quad (7)$$

$$\Pi = p^V \wedge \tilde{V} - \frac{M^V}{c} = p^V \wedge \tilde{V} + \star \left( \frac{m^V}{c} \wedge \tilde{V} \right) \quad (8)$$

$$\mathcal{J} = \frac{-J^V}{c} \wedge \tilde{V} - \rho^V \# 1 \quad (9)$$

where  $i_V e^V = 0$ ,  $i_V d^V = 0$ ,  $i_V p^V = 0$ ,  $i_V J^V = 0$ ,  $i_V B^V = 0$ ,  $i_V H^V = 0$ ,  $i_V M^V = 0$ ,  $i_V \rho^V \# 1 = 0$ . In these expressions  $e^V$ ,  $E^V$ , denote a (time-dependent) spatial electric field 1-form and 2-form respectively;  $d^V$ ,  $D^V$ , denote a (time-dependent) spatial electric displacement field 1-form and 2-form respectively;  $b^V$ ,  $B^V$ , denote a (time-dependent) spatial magnetic induction field 1-form and 2-form respectively;  $h^V$ ,  $H^V$ , denote a (time-dependent) spatial magnetic field field 1-form and 2-form respectively;  $p^V$ ,  $P^V$ , denote a (time-dependent) spatial electric polarisation field field 1-form and 2-form respectively;  $m^V$ ,  $M^V$ , denote a (time-dependent) spatial magnetic polarisation field field 1-form and 2-form respectively;  $J^V$  denotes a (time-dependent) spatial 3-current 2-form and  $\rho^V$  denotes a (time-dependent) spatial electric charge density 0-form. It should be stressed that these decompositions are defined for arbitrary observer fields, including those describing accelerated frames.

In Minkowski spacetime there exist global coordinates  $t, x, y, z$  in which the above g-orthonormal cobasis takes the form:

$$e^0 = c dt, e^1 = dx, e^2 = dy, e^3 = dz \quad (10)$$

The history of an inertial observer is then part of an integral curve of the vector field

$$U = \frac{1}{c} \partial_t \quad (11)$$

and

$$\tilde{U} = -cdt \quad (12)$$

In such a spacetime coordinate system the inertial frame time rate of change of any time dependent  $p$ -form  $\alpha$  is defined as the Lie derivative  $c\mathcal{L}_U \alpha$  and denoted  $\dot{\alpha}$ .

If one makes the orthogonal decompositions above with respect to such an inertial (laboratory) frame the Maxwell system (2) yields:

$$\# \underline{d} \# b^U = 0 \quad \text{or} \quad \nabla \cdot \mathbf{B}^U = 0 \quad (13)$$

$$\# \underline{d} e^U = -\dot{b}^U \quad \text{or} \quad \nabla \times \mathbf{E}^U = -\dot{\mathbf{B}}^U \quad (14)$$

$$\# \underline{d} \# d^U = \rho^U \quad \text{or} \quad \nabla \cdot \mathbf{D}^U = \rho^U \quad (15)$$

$$\# \underline{d} h^U = \dot{d}^U + j^U \quad \text{or} \quad \nabla \times \mathbf{H}^U = -\dot{\mathbf{D}}^U + \mathbf{J}^U \quad (16)$$

where  $j^U = \# J^U$  and  $\dot{\alpha} = \mathcal{L}_{\partial_t} \alpha$ . The bold-face characters  $\mathbf{E}^U, \mathbf{D}^U, \mathbf{B}^U, \mathbf{H}^U, \mathbf{J}^U$  refer to the traditional Cartesian components of the associated 3-vector fields in the inertial (laboratory) frame  $U$ . The relation  $G = \epsilon_0 F + \Pi$  yields

$$\mathbf{D}^U = \epsilon_0 \mathbf{E}^U + \mathbf{P}^U \quad (17)$$

$$\mathbf{H}^U = \frac{1}{\mu_0} \mathbf{B}^U + \mathbf{M}^U \quad (18)$$

and substituting these into the Maxwell equations (14,16) gives,

$$\dot{h}^U = -\frac{1}{\mu_0} \# \underline{d} e^U + \dot{m}^U \quad \text{or} \quad \dot{\mathbf{H}}^U = -\frac{1}{\mu_0} \nabla \times \mathbf{E}^U + \dot{\mathbf{M}}^U \quad (19)$$

$$\dot{e}^U = \frac{1}{\epsilon_0} (\# \underline{d} h^U - j^U - \dot{p}^U) \quad \text{or} \quad \dot{\mathbf{E}}^U = \frac{1}{\epsilon_0} (\nabla \times \mathbf{H}^U - \mathbf{J}^U - \dot{\mathbf{P}}^U) \quad (20)$$

### 3. Covariant Constitutive Models

To set the models to be discussed in context it is worth recalling the simplest covariant constitutive model describing a non-dispersive, non-hysteretic, homogeneous, isotropic linear material with arbitrary 4-velocity  $V$ :

$$G = \epsilon_0 \left( \epsilon_r - \frac{1}{\mu_r} \right) i_V F \wedge \tilde{V} + \frac{\epsilon_0}{\mu_r} F \quad (21)$$

where  $\epsilon_r$  and  $\mu_r$  are dimensionless constants. In the co-moving frame  $V$  these yield

$$\mathbf{D}^V = \epsilon_0 \epsilon_r \mathbf{E}^V, \quad \mathbf{H}^V = \frac{1}{\mu_0 \mu_r} \mathbf{B}^V \quad (22)$$

with electric displacement in the co-moving frame proportional to electric field in that frame and magnetic field in the co-moving frame proportional to magnetic induction in that frame. If  $G$  and  $F$  in (21) are decomposed with respect to different frames (possibly accelerating) then the relations between the electric and magnetic fields in different frames are different and involve the instantaneous relative 3-velocities between the frames. This is often referred to as a *motion induced* magneto-electric effect.

Since  $\Pi = G - \epsilon_0 F$ ,  $\star((i_V \star G) \wedge \tilde{V}) = -(1 + \tilde{V} \wedge i_V)G$ , and  $i_V G = \epsilon_0 \epsilon_r i_V F$  equation (21) may be written as an algebraic local relation between  $\Pi$ ,  $F$ ,  $G$  and  $V$ .

$$\Pi = \epsilon_0 (\epsilon_r - 1) i_V F \wedge \tilde{V} + (\mu_r - 1) \star((i_V \star G) \wedge \tilde{V}) \quad (23)$$

If the medium is conducting a constitutive relation for a conductivity current is required. In an inertial frame  $U$  with local coordinates  $t, x, y, z$  suppose the total current density 1-form is

$$j^U(x, y, z, t) = j_{cond}^U(x, y, z, t) + j_{ext}^U(x, y, z, t) \quad (24)$$

where  $j_{ext}^U(x, y, z, t)$  denotes a prescribed external current. A simple isotropic Ohmic conductivity current arises from the temporal non-local relation:

$$j_{cond}^U(x, y, z, t) = \int_{-\infty}^t \kappa(t' - t) e^U(x, y, z, t') dt' \quad (25)$$

since its Fourier transform with respect to  $t$  yields:

$$\hat{j}_{cond}^U(x, y, z, \omega^U) = \sigma(\omega^U) \hat{e}^U(x, y, z, \omega^U) \quad (26)$$

where the scalar conductivity  $\sigma(\omega^U)$  is the Fourier transform of the spatially homogeneous scalar  $\kappa(t)$ ,  $\hat{j}_{cond}^U$  is the Fourier transform of  $j_{cond}^U$  and  $\hat{e}^U$  is the Fourier transform of  $e^U$ . A spacetime model therefore requires a model for  $\kappa$ . In practice one finds data for  $\sigma(\omega^U)$  over some restricted range of  $\omega^U$  which is rarely sufficient, in general, to fully re-construct  $\kappa$  by Fourier inversion. However in circumstances where the electromagnetic fields have Fourier components dominantly in the frequency range of relevance the approximation

$$j_{cond}^U(x, y, z, t) \simeq \sigma(\omega^U) e^U(x, y, z, t) \quad (27)$$

often suffices. In this approximation  $\sigma(\omega^U)$  is regarded as a homogeneous scalar field in space.

To extend these models to a spatially non-dispersive but anisotropic hysteretic, purely ferromagnetic or ferroelectric medium it is necessary to accommodate non-local differential constitutive relations with memory. Such a medium has a preferred material eigen-basis determined by its ferromagnetic or ferroelectric susceptibility tensor. This determines the direction in space of the induced magnetic or electric polarisation relative to the direction of an applied field. A model for a uni-dimensional, rate-independent ferromagnetic medium was constructed by Coleman and Hodgdon [5], [6]. In the notation of this article, with inertial spatial field components taken in a preferred  $\underline{\mathbf{g}}$ -ortho-normal eigen-basis it took the form:

$$\dot{b}_j^U = \alpha_j \left| \dot{h}_j^U \right| (f_j(h_j^U) - b_j^U) + \dot{h}_j^U g_j(h_j^U) \quad \text{for } j = 1, 2, 3 \quad (28)$$

where the magnetic field  $h_j^U$  and the magnetic induction  $b_j^U$  were continuous slowly varying real-valued functions of time with piecewise continuous derivatives,  $\dot{h}_j^U$ ,  $\dot{b}_j^U$ , positive constants  $\alpha_j$  and  $f_j$  and  $g_j$  specified real-valued functions on the real line. Following the approach described in the introduction one seeks a covariant extension applicable to ferroelectrics and ferromagnets that reduces for slowly varying electric or magnetic fields in any inertial frame to a similar type of Coleman-Hodgdon model. Since such a moving ferromagnet can acquire an induced electric polarisation and such a moving ferroelectric can acquire a magnetic polarisation, a more general inertial model can be constructed that exhibits a hysteretic intrinsic magneto-electric response to slowly varying fields in any *inertial* frame. Needless to say there can be no unique extension to accelerating media.

To construct such covariant models we introduce a number of rank (1,1)  $V$ -orthogonal spatial tensors *on spacetime*. Each such tensor  $\chi$  maps 1-forms to 1-forms and satisfies  $\chi(V, -) = 0$  and  $\chi(-, \tilde{V}) = 0$  for any time-like vector field  $V$ . It has an associated  $\underline{\mathbf{g}}$ -ortho-normal eigenbasis  $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$  and in this basis

$$\chi = \sum_{j=1}^3 \chi_j \tilde{\mathcal{E}}_j \otimes \mathcal{E}_j \quad (29)$$

In terms of four (in general distinct) such spatial tensors an *intrinsic magneto-electric* hysteretic model that has the required inertial behaviour takes the form

$$\begin{aligned} \nabla_V \Pi = & \chi_{pe}^V (\nabla_V i_V F) \wedge \tilde{V} + \chi_{ph}^V (\nabla_V i_V \star G) \wedge \tilde{V} - \\ & \star \left\{ \chi_{mh}^V (\nabla_V i_V \star G) \wedge \tilde{V} + \chi_{me}^V (\nabla_V i_V F) \wedge \tilde{V} \right\} \end{aligned} \quad (30)$$

where  $\nabla_V$  denotes a covariant derivative with respect to  $V$  and the  $V$ -orthogonal spatial tensors  $\chi_Q^V$  satisfy  $\chi_Q^V(V, -) = 0$  and  $\chi_Q^V(-, \tilde{V}) = 0$  with  $Q=pe, ph, mh, me$ . They characterize the hysteretic properties of the medium. If all these tensors are non-zero for all  $V$  then the material is totally intrinsically magneto-electric.

Since  $\nabla_V \Pi \neq \nabla_V \star \Pi$  when the (Levi-Civita) 4-acceleration  $\nabla_V V$  of the medium is non-zero, there exists a physically distinct “dual” model

$$\begin{aligned} \nabla_V \star \Pi &= \chi_{mh}^V (\nabla_V i_V \star G) \wedge \tilde{V} + \chi_{me}^V (\nabla_V i_V F) \wedge \tilde{V} + \\ &\star \left\{ \chi_{pe}^V (\nabla_V i_V F) \wedge \tilde{V} + \chi_{ph}^V (\nabla_V i_V \star G) \wedge \tilde{V} \right\} \end{aligned} \quad (31)$$

that has the same behavior for media at rest in all inertial frames.

The hysteretic behavior of the medium is determined by parameterising the components of each spatial tensor in its associated material eigenbasis. Thus with

$$\chi_Q^V = \sum_{j=1}^3 \chi_{Q,j}^V \tilde{\mathcal{E}}_j^V \otimes \mathcal{E}_j^V \quad (32)$$

the scalar components  $\chi_{Q,j}^V (e_j^V, p_j^V, h_j^V, m_j^V)$  are written

$$\chi_{pe,j}^V = \Psi_{pe,j}^V \left[ \kappa_j^{pe} \operatorname{sgn}(\Psi_{pe,j}^V) + \theta_j^{pe} \operatorname{sgn} \left( i_{\mathcal{E}_j^V} \nabla_V e^V \right) \right], \quad (33)$$

$$\chi_{ph,j}^V = \Psi_{ph,j}^V \left[ \kappa_j^{ph} \operatorname{sgn}(\Psi_{ph,j}^V) + \theta_j^{ph} \operatorname{sgn} \left( i_{\mathcal{E}_j^V} \nabla_V \frac{h^V}{c} \right) \right], \quad (34)$$

$$\chi_{mh,j}^V = \Psi_{mh,j}^V \left[ \kappa_j^{mh} \operatorname{sgn}(\Psi_{mh,j}^V) + \theta_j^{mh} \operatorname{sgn} \left( i_{\mathcal{E}_j^V} \nabla_V \frac{h^V}{c} \right) \right], \quad (35)$$

$$\chi_{me,j}^V = \Psi_{me,j}^V \left[ \kappa_j^{me} \operatorname{sgn}(\Psi_{me,j}^V) + \theta_j^{me} \operatorname{sgn} \left( i_{\mathcal{E}_j^V} \nabla_V e^V \right) \right] \quad (36)$$

with  $e^V = i_V F$ ,  $h^V = -ci_V \star G$  for  $j = 1, 2, 3$  and constants  $\kappa_j^Q, \theta_j^Q \in \mathbb{R}^+$ .

The  $\operatorname{sgn}$  function defined by

$$\operatorname{sgn}(z) = z/|z| = \begin{cases} +1 & \text{for } z > 0 \\ 0 & \text{for } z = 0 \\ -1 & \text{for } z < 0 \end{cases} \quad (37)$$

accommodates the branching during the hysteretic process  $\ddagger$ ,

$$\Psi_{pe,j}^V = \epsilon_0 (\alpha_j^{pe} f_j^{pe}(e_j^V) - \xi_j^{pe} p_j^V) \quad (38)$$

$$\Psi_{ph,j}^V = - \left( \alpha_j^{ph} f_j^{ph}(h_j^V) - \xi_j^{ph} p_j^V \right) \quad (39)$$

$$\Psi_{me,j}^V = \epsilon_0 (\alpha_j^{me} f_j^{me}(e_j^V) + \xi_j^{me} m_j^V) \quad (40)$$

$$\Psi_{mh,j}^V = - \left( \alpha_j^{mh} f_j^{mh}(h_j^V) + \xi_j^{mh} m_j^V \right) \quad (41)$$

with constants  $\alpha_j^Q, \xi_j^Q \in \mathbb{R}^+$  and frame-dependent scalars

$$e_j^V = i_{\mathcal{E}_j^V} i_V F, \quad e_j^V = i_{\mathcal{E}_j^V} i_V F, \quad (42)$$

$$h_j^V = -ci_{\mathcal{E}_j^V} i_V \star G, \quad h_j^V = -ci_{\mathcal{E}_j^V} i_V \star G, \quad (43)$$

$$p_j^V = i_{\mathcal{E}_j^V} i_V \Pi, \quad m_j^V = -ci_{\mathcal{E}_j^V} i_V \star \Pi \quad (44)$$

$\ddagger$  The factors of  $\epsilon_0$  imply that  $\Psi_{pe,j}^V/\epsilon_0, \Psi_{ph,j}^V, \Psi_{me,j}^V/\epsilon_0, \Psi_{mh,j}^V$  are dimensionless.

The real valued function  $f_j^Q$  characterizes the details of the hysteretic process. For the explicit computation below it is taken to have the form

$$f_j^Q(z) = \tanh(\beta_j^Q z) \quad (45)$$

with  $\beta_j^Q \in \mathbb{R}^+$  and where  $\lim_{z \rightarrow \pm\infty} f_j^Q(z) = \pm 1$ . The parameters  $\alpha_j^Q, \beta_j^Q, \xi_j^Q, \kappa_j^Q, \theta_j^Q$  must be motivated by data in the laboratory frame.

The use of the covariant derivative in (30) ensures that a hysteretic process driven by a harmonic external field can give rise to a autonomous differential equation on any *smooth branch* of a hysteresis loop as described in the next section. In any inertial frame  $U$  both (30) and (31) yield the coupled system of partial differential equations:

$$\dot{p}_j^U = \chi_{pe,j}^U (\dot{e}_j^U) - \frac{1}{c} \chi_{ph,j}^U (\dot{h}_j^U) \quad (46)$$

$$\dot{m}_j^U = \chi_{mh,j}^U (\dot{h}_j^U) - c \chi_{me,j}^U (\dot{e}_j^U) \quad (47)$$

i.e.

$$\begin{aligned} \dot{p}_j^U &= \kappa_j^{pe} \dot{e}_j^U |\Psi_{pe,j}^U| + \theta_j^{pe} |\dot{e}_j^U| \Psi_{pe,j}^U - \\ &\quad \frac{1}{c} \left[ \kappa_j^{ph} \dot{h}_j^U |\Psi_{ph,j}^U| + \theta_j^{ph} |\dot{h}_j^U| \Psi_{ph,j}^U \right] \end{aligned} \quad (48)$$

$$\begin{aligned} \dot{m}_j^U &= \kappa_j^{mh} \dot{h}_j^U |\Psi_{mh,j}^U| + \theta_j^{mh} |\dot{h}_j^U| \Psi_{mh,j}^U - \\ &\quad c \left[ \kappa_j^{me} \dot{e}_j^U |\Psi_{me,j}^U| + \theta_j^{me} |\dot{e}_j^U| \Psi_{me,j}^U \right] \end{aligned} \quad (49)$$

where  $|\dot{e}_j^U| \equiv |i_{\mathcal{E}^V} \nabla_V e^V|$  and  $|\dot{h}_j^U| \equiv |i_{\mathcal{E}^V} \nabla_V \frac{h^V}{c}|$ .

#### 4. A Particular Model at Rest in an Inertial (Laboratory) Frame

We consider the special case of a non-magneto-electric medium (30) where  $\chi_{ph}^V = 0$  and  $\chi_{me}^V = 0$  and:

$$\nabla_V \Pi = \chi_{pe}^V (\nabla_V i_V F) \wedge \tilde{V} - \star \left( \chi_{mh}^V (\nabla_V i_V \star G) \wedge \tilde{V} \right) \quad (50)$$

In terms of spatial fields it follows from (8) that (50) may be written,

$$\begin{aligned} \nabla_V p^V \wedge \tilde{V} + p^V \wedge \nabla_V \tilde{V} - \nabla_V \left[ \star \left( \frac{m^V}{c} \wedge \tilde{V} \right) \right] = \\ \chi_{pe}^V (\nabla_V e^V) \wedge \tilde{V} + \star \left( \chi_{mh}^V \left( \nabla_V \frac{h^V}{c} \right) \wedge \tilde{V} \right) \end{aligned} \quad (51)$$

For the medium at rest in the inertial frame  $U = \frac{1}{c} \partial_t$  one has  $V = U$  and equation (51) becomes

$$\dot{p}^U \wedge \tilde{U} + \star \left( \frac{\dot{m}^U}{c} \wedge \tilde{U} \right) = \chi_{pe}^U (\dot{e}^U) \wedge \tilde{U} + \star \left( \chi_{mh}^U \left( \frac{\dot{h}^U}{c} \right) \wedge \tilde{U} \right) \quad (52)$$

where for any spatial  $p$ -form  $\xi^U$

$$\dot{\xi}^U \equiv \nabla_{\partial_t} \xi^U \quad (53)$$

since the (laboratory) inertial cobasis is parallel. Taking components in the direction of  $U$  and its orthogonal subspace yields

$$\dot{p}^U = \chi_{pe}^U (\dot{e}^U) \quad (54)$$

$$\dot{m}^U = \chi_{mh}^U (\dot{h}^U) \quad (55)$$



From (48) to (49) one has in the  $U$  frame decoupled equations for the hysteretic electric and magnetic polarisation fields in the medium:

$$\dot{p}_j^U = \kappa_j^{pe} \dot{e}_j^U |\Psi_{pe,j}^U| + \theta_j^{pe} |\dot{e}_j^U| \Psi_{pe,j}^U \quad (56)$$

$$\dot{m}_j^U = \kappa_j^{mh} \dot{h}_j^U |\Psi_{mh,j}^U| + \theta_j^{mh} |\dot{h}_j^U| \Psi_{mh,j}^U \quad (57)$$

where  $p^U = p_1^U dx + p_2^U dy + p_3^U dz$  etc.

Over an *arbitrary* time interval each of the above differential equations is a non-autonomous evolution equation at each point in the medium describing induced polarisations as a function of time. However in certain time domains their evolution is controlled by autonomous ordinary differential equations. When the drive fields are harmonic in time the resulting solutions to such equations may exhibit a limit cycle, (often identified as a hysteresis loop) composed of piecewise smooth line segments. For such cycles controlled by (56) the location and number of piecewise smooth segments in the cycle (branches) are determined by the location and number of simultaneous zeroes of  $\dot{e}_j^U$  and  $p_j^U$  in the cycle. Similarly in cycles controlled by (57) branches are determined by the location and number of simultaneous zeroes of  $\dot{h}_j^U$  and  $m_j^U$  in the cycle. The choice of parameters in the function  $f_j^Q$  determines such locations and also the degree of induced saturation in each branch of a limit cycle during the process. When  $\dot{e}_j^U$  and  $p_j^U$  have more general time-dependences the above equations give rise to solutions that may exhibit hysteretic loci containing self-intersections and/or no limit cycle.

Thus at each spatial point, (56) can be written

$$\frac{\dot{p}_j^U}{\dot{e}_j^U} = \frac{dp_j^U}{de_j^U} = \kappa_j^{pe} \Psi_{pe,j}^U \text{sgn}(\dot{e}_j^U) + \theta_j^{pe} \text{sgn}(\dot{e}_j^U) \Psi_{pe,j}^U \quad (58)$$

For each of the possible combinations of  $\text{sgn}(\dot{e}_j^U)$  and  $\text{sgn}(\Psi_{pe,j}^U)$ , (58) yields a first-order differential equation for  $p_j^U(e)$  with branched solutions through any point  $(P_0^U, E_0^U)$ . If one denotes  $\text{sgn}(\dot{e}_j^U)$  by  $S_E$  and  $\text{sgn}(\Psi_{pe,j}^U)$  by  $S_\Psi$  and furthermore assumes that parameters are chosen so that  $S_\Psi$  is constant on a particular branch a solution through this point can be written

$$p_j^U(e_j^U; P_0^U, E_0^U, S_\Psi, S_E) = P_0^U e^{-\eta_j^{pe} \xi_j^{pe} (e_j^U - E_0^U)} + e^{-\eta_j^{pe} \xi_j^{pe} e_j^U} \eta_j^{pe} \alpha_j^{pe} \int_{E_0^U}^{e_j^U} f_j^{pe}(\Upsilon) e^{\eta_j^{pe} \xi_j^{pe} \Upsilon} d\Upsilon \quad (59)$$

where  $\eta_j^{pe} = \epsilon_0 (\kappa_j^{pe} S_\Psi + \theta_j^{pe} S_E)$ . Although  $S_E$  is +1 when  $(\dot{e}_j^U)$  is increasing with time or -1 when  $(\dot{e}_j^U)$  is decreasing with time the sign of  $S_\Psi$  will in general depend on the state  $p_j^U(e_j)$ . Furthermore when the parameters  $\alpha_j^Q, \beta_j^Q, \xi_j^Q, \kappa_j^Q$  and  $\theta_j^Q$  vary inhomogeneously with position in the medium this state will also depend explicitly on position. In such situations an analytic solution is no longer in general possible. Analogous solutions may be written for (57) and when  $\kappa_j^{mh} = 0$  such branched solutions coincide with those in [6] when  $\theta_j^{mh}$  is constant.

## 5. A Particular Model Rotating in an Inertial (Laboratory) Frame

In this section the model is applied to a rigid uniformly axially rotating cylinder of radius  $R$  in an external time harmonic electromagnetic field in an effort to see

the order of magnitude of effects on a hysteretic process in an accelerating medium. The differential constitutive relations are appended to the Maxwell system and the resulting coupled partial differential system analyzed numerically. The system is wholly enclosed in a rectangular 3-dimensional computational domain that simulates the rotating cylinder in a perfectly conducting cavity. The choice of parameters in the model is motivated by establishing results that may be compared with hysteretic behaviour of a medium at rest in a low frequency electromagnetic environment. The modern measurement of ferromagnetic and ferroelectric hysteresis in such situations is an experimental art and owes much to modern digital and piezoelectric technology. Needless to say great care is exercised to accommodate effects such as sample geometry, material electrical conductivity variations with frequency, thermal drift instabilities and material spatial inhomogeneities [8]. In this article we will use a particular hysteresis measurement [9] more as a guide to a reasonable parameter set for our model rather than a targeted fit to a specific material specimen.

The model is applied to a 1.36m radius cylinder, rotating at an angular speed 1497rpm inside a perfectly conducting cavity with sides of lengths 5.45m×5.45m×2.18m. The cavity fields are driven by an assembly of externally prescribed currents inside the cavity at 250Hz, well below the lowest natural electromagnetic mode of such a cavity. For the above angular speed and cavity dimension a lattice  $20 \times 20 \times 8$  discretization (using a fast workstation) permits exploration of the hysteretic processes in the rotating medium when the above differential equations are coupled to the macroscopic Maxwell equations. In this approach all components of spatial electromagnetic fields and spatial polarisations are with respect to the laboratory frame  $U$  and interest is directed to how these fields evolve parametrically with time and hence with each other in such a rotating medium.

We consider a purely ferroelectric medium that is magnetically inert in its rest frame. This is simply achieved by setting each  $\chi_{mh,j}^V$  above to zero. The first part of the computation involves the determination of the material eigen-basis in the rotating medium. To this end it is natural to adopt a polar Minkowski cobasis rather than a Cartesian one. With the cylinder rotating about the laboratory  $z$ -axis with uniform angular speed  $\Omega$  radians per second the Minkowski metric in terms of a cylindrical polar basis  $e^0 = cdt$ ,  $e^1 = dr$ ,  $e^2 = r d\phi$ ,  $e^3 = dz$ , becomes

$$\mathbf{g} = -c^2 dt \otimes dt + r^2 d\phi \otimes d\phi + dr \otimes dr + dz \otimes dz \quad (60)$$

where

$$r^2 = x^2 + y^2, \quad x = r \cos \phi, \quad y = r \sin \phi \quad (61)$$

$$\partial_\phi = x \partial_y - y \partial_x, \quad \partial_r = \frac{1}{r} (x \partial_x + y \partial_y) \quad (62)$$

$$d\phi = \frac{1}{r^2} (x dy - y dx), \quad dr = \frac{1}{r} (x dx + y dy) \quad (63)$$

The bulk 4-velocity field of the cylinder is,

$$V = \frac{\gamma}{c} (\partial_t + \Omega \partial_\phi) \quad \text{with} \quad \gamma = \left(1 - \frac{r^2 \Omega^2}{c^2}\right)^{-\frac{1}{2}} \quad (64)$$

Since  $r\Omega \ll c$  for  $r \leq R$ ,

$$V \simeq \frac{1}{c} \partial_t + \frac{\Omega}{c} (x \partial_y - y \partial_x) \quad (65)$$

$$\tilde{V} \simeq -cdt + \frac{\Omega}{c} (x dy - y dx) \quad (66)$$

and  $\mathbf{g}(V, V) \simeq r^2 \frac{\Omega^2}{c^2} - 1 \approx -1$ . By symmetry the material basis may be written

$$\tilde{\mathcal{E}}_1^V = \tilde{W}^V, \quad \tilde{\mathcal{E}}_2^V = dr, \quad \tilde{\mathcal{E}}_3^V = dz \quad (67)$$

$$\mathcal{E}_1^V = W^V, \quad \mathcal{E}_2^V = \partial_r, \quad \mathcal{E}_3^V = \partial_z \quad (68)$$

where

$$W^V = \hat{\alpha} \partial_t + \hat{\beta} \partial_\phi \quad (69)$$

$$\tilde{W}^V = -c^2 \hat{\alpha} dt + r^2 \hat{\beta} d\phi \quad (70)$$

for constants  $\hat{\alpha}$  and  $\hat{\beta}$ . Imposing the orthogonality conditions  $\mathbf{g}(W^V, V) = 0$  and  $\mathbf{g}(W^V, W^V) = 0$  yields

$$\hat{\alpha} = \frac{r\Omega}{c^2}, \quad \hat{\beta} = \frac{1}{r} \quad (71)$$

All fields can now be projected into the laboratory  $U$  frame, terms of order  $(r\Omega)^2/c^2$  removed and re-expressed in terms of the Minkowski cobasis in  $t, x, y, z$  coordinates.

Since  $\nabla_V \tilde{U} = 0$ ,  $\nabla_U \tilde{U} = 0$  the left hand side of (50) is

$$\begin{aligned} \nabla_V \Pi &= \nabla_{U + \frac{\Omega}{c}(x\partial_y - y\partial_x)} \left( p^U \wedge \tilde{U} - \frac{M^U}{c} \right) \\ &= \left( U p^U + \frac{\Omega}{c} (x\partial_y - y\partial_x) p^U \right) \wedge \tilde{U} - \frac{1}{c} U M^U - \\ &\quad \frac{\Omega}{c^2} (x\partial_y - y\partial_x) M^U \end{aligned} \quad (72)$$

and the right hand side of (50) is

$$\chi_{pe}^V (\nabla_V i_V F) \wedge \tilde{V} = \chi_{pe}^V (\nabla_V e^V) \wedge \left( \tilde{U} + \frac{\Omega}{c} (x dy - y dx) \right) \quad (73)$$

Here the six components of (50) in the basis (10) yield the following coupled PDE system for the functions  $p_x^U(t, x, y, z)$ ,  $p_y^U(t, x, y, z)$ ,  $p_z^U(t, x, y, z)$ ,  $m_x^U(t, x, y, z)$ ,  $m_y^U(t, x, y, z)$ ,  $m_z^U(t, x, y, z)$ :

*Electric Sector*

$$\begin{aligned} \dot{p}_x^U + \Omega (x\partial_y - y\partial_x) p_x^U &= \frac{1}{r^2} \{ y^2 \chi_{pe,1}^V + x^2 \chi_{pe,2}^V \} \cdot \\ &\quad \left\{ \dot{e}_x^U + \Omega x \dot{b}_z^U + \Omega (x\partial_y - y\partial_x) e_x^U \right\} + \\ &\quad \frac{1}{r^2} xy \{ -\chi_{pe,1}^V + \chi_{pe,2}^V \} \cdot \\ &\quad \left\{ \dot{e}_y^U + \Omega y \dot{b}_z^U + \Omega (x\partial_y - y\partial_x) e_y^U \right\} \end{aligned} \quad (74)$$

$$\begin{aligned} \dot{p}_y^U + \Omega (x\partial_y - y\partial_x) p_y^U &= \frac{1}{r^2} \{ x^2 \chi_{pe,1}^V + y^2 \chi_{pe,2}^V \} \cdot \\ &\quad \left\{ \dot{e}_{ep,y}^U + \Omega y \dot{b}_z^U + \Omega (x\partial_y - y\partial_x) e_y^U \right\} + \\ &\quad \frac{1}{r^2} xy \{ -\chi_{pe,1}^V + \chi_{pe,2}^V \} \cdot \\ &\quad \left\{ \dot{e}_x^U + \Omega x \dot{b}_z^U + \Omega (x\partial_y - y\partial_x) e_x^U \right\} \end{aligned} \quad (75)$$

$$\dot{p}_z^U + \Omega (x\partial_y - y\partial_x) p_z^U = \chi_{pe,3}^V \left\{ \dot{e}_z^U - \Omega [x\dot{b}_x^U + y\dot{b}_y^U] + \Omega (x\partial_y - y\partial_x) e_z^U \right\} \quad (76)$$

*Magnetic Sector*

$$\dot{m}_x^U + \Omega (x\partial_y - y\partial_x) m_x^U = x\Omega\chi_{pe,3}^V \dot{e}_z^U \quad (77)$$

$$\dot{m}_y^U + \Omega (x\partial_y - y\partial_x) m_y^U = y\Omega\chi_{pe,3}^V \dot{e}_z^U \quad (78)$$

$$\dot{m}_z^U + \Omega (x\partial_y - y\partial_x) m_z^U = -\Omega\chi_{pe,2}^V \{x\dot{e}_x^U + y\dot{e}_y^U\} \quad (79)$$

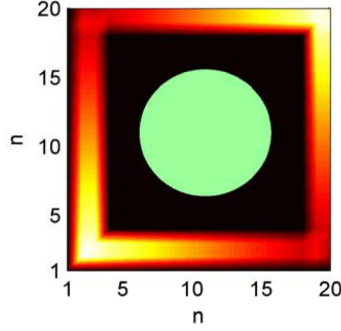
with

$$\begin{aligned} \chi_{pe,1}^V &= \epsilon_0 \left\{ \alpha_1^{pe} f_1^{pe} \left( \frac{xe_y^U - ye_x^U}{r} \right) + \xi_1^{pe} \frac{yp_x^U - xp_y^U}{r} \right\} \cdot \\ &\quad \left\{ \kappa_1^{pe} \operatorname{sgn} \left( \epsilon_0 \left\{ \alpha_1^{pe} f_1^{pe} \left( \frac{xe_y^U - ye_x^U}{r} \right) + \xi_1^{pe} \frac{yp_x^U - xp_y^U}{r} \right\} \right) + \right. \\ &\quad \left. \theta_1^{pe} \operatorname{sgn} \left( \frac{-y\dot{e}_x^U + x\dot{e}_y^U - \Omega y (x\partial_y - y\partial_x) e_x^U + \Omega x (x\partial_y - y\partial_x) e_y^U}{r} \right) \right\} \\ \chi_{pe,2}^V &= \epsilon_0 \left\{ \alpha_2^{pe} f_2^{pe} \left( \frac{xe_x^U + ye_y^U}{r} + \Omega r b_z^U \right) - \xi_2^{pe} \left[ \frac{xp_x^U + yp_y^U}{r} + \frac{\Omega r}{c^2} m_z^U \right] \right\} \cdot \\ &\quad \left\{ \kappa_2^{pe} \operatorname{sgn} \left( \epsilon_0 \left\{ \alpha_2^{pe} f_2^{pe} \left( \frac{xe_x^U + ye_y^U}{r} + \Omega b_z^U r \right) - \xi_2^{pe} \left[ \frac{xp_x^U + yp_y^U}{r} + \frac{\Omega r}{c^2} m_z^U \right] \right\} \right) + \right. \\ &\quad \left. \theta_2^{pe} \operatorname{sgn} \left( \frac{x\dot{e}_x^U + y\dot{e}_y^U + \Omega r^2 \dot{b}_z^U + \Omega x (x\partial_y - y\partial_x) e_x^U + \Omega y (x\partial_y - y\partial_x) e_y^U}{r} \right) \right\} \\ \chi_{pe,3}^V &= \epsilon_0 \left\{ \alpha_3^{pe} f_3^{pe} (e_z^U - \Omega [xb_x^U + yb_y^U]) - \xi_3^{pe} \left[ p_z^U - \Omega \frac{xm_x^U + ym_y^U}{c^2} \right] \right\} \cdot \\ &\quad \left\{ \kappa_3^{pe} \operatorname{sgn} \left( \epsilon_0 \left\{ \alpha_3^{pe} f_3^{pe} (e_z^U - \Omega [xb_x^U + yb_y^U]) - \xi_3^{pe} \left[ p_z^U - \Omega \frac{xm_x^U + ym_y^U}{c^2} \right] \right\} \right) + \right. \\ &\quad \left. \theta_3^{pe} \operatorname{sgn} \left( \dot{e}_z^U - \Omega [y\dot{b}_y^U + x\dot{b}_x^U] + \Omega (x\partial_y - y\partial_x) e_z^U \right) \right\} \end{aligned}$$

and

$$\chi_{mh,1}^V = 0, \quad \chi_{mh,2}^V = 0, \quad \chi_{mh,3}^V = 0 \quad (80)$$

The presence of the discontinuous sgn functions in these equations is responsible for the branched structure of their solutions.



**Figure 1.** The computational domain is the space inside a perfectly conducting rectangular cavity with sides with lengths  $5.45\text{m} \times 5.45\text{m} \times 2.18\text{m}$ . Figure 1 shows an  $x-y$  cross-section of the cavity at constant  $z$ . The computation is done on a  $20 \times 20 \times 8$  3-dimensional lattice in the cavity. The fields in the cavity are driven by a spatially divergence-free time-harmonic current density that flows parallel to the pairs of  $x-y$ ,  $y-z$  and  $z-x$  interior faces of the cavity without penetrating the cylindrical ferroelectric material with radius  $1.36\text{m}$ , centered at the origin, oriented with its axis of rotational symmetry along the  $z$ -axis of the cavity. A time-harmonic current density is taken proportional to the spatial vector  $(z, z, x)$  modulated to have support that excludes the cylinder. In the figure the gradation of shading from dark to light around the  $x-y$  perimeter depicts, from minimum to maximum, the magnitude of the current density in the cross section at a particular instant of time.

Before coupling these constitutive equations to the Maxwell equations a further simplification is adopted. Suppose the medium has a uni-axial symmetry direction that can be aligned along the laboratory  $z$ -axis when it is at rest in the laboratory frame. Since the rotary motion is transverse to this axis this implies:

$$\begin{aligned} \chi_{pe,1}^V &= 0, \quad \chi_{pe,2}^V = 0 \\ \chi_{pe,3}^V &= \epsilon_0 \{ \alpha_3^{pe} f_3^{pe} (e_3^V) - \xi_3^{pe} p_3^V \}. \end{aligned} \quad (81)$$

$$\begin{aligned} &\{ \kappa_3^{pe} \text{sgn} (\epsilon_0 [\alpha_3^{pe} f_3^{pe} (e_3^V) - \xi_3^{pe} p_3^V]) + \\ &\theta_3^{pe} \text{sgn} (\dot{e}_z^U - \Omega [y \dot{b}_y^U + x \dot{b}_x^U] + \Omega (x \partial_y - y \partial_x) e_z^U) \} \end{aligned} \quad (82)$$

The equations (74)-(80) then simplify to

$$\dot{p}_x^U = \Omega (y \partial_x - x \partial_y) p_x^U \quad (83)$$

$$\dot{p}_y^U = \Omega (y \partial_x - x \partial_y) p_y^U \quad (84)$$

$$\begin{aligned} \dot{p}_z^U &= \Omega (y \partial_x - x \partial_y) p_z^U + \epsilon_0 \{ \alpha_3^{pe} f_3^{pe} (e_3^V) - \xi_3^{pe} p_3^V \} \cdot \\ &\{ \kappa_3^{pe} \mathcal{E}_3^V \text{sgn} (\epsilon_0 [\alpha_3^{pe} f_3^{pe} (e_3^V) - \xi_3^{pe} p_3^V]) + \theta_3^{pe} |\mathcal{E}_3^V| \} \end{aligned} \quad (85)$$

$$\begin{aligned} \dot{m}_x^U &= \Omega (y \partial_x - x \partial_y) m_x^U + \\ &+ x \Omega \epsilon_0 \kappa_3^{pe} |\alpha_3^{pe} f_3^{pe} (e_z^U) - \xi_3^{pe} p_z^U| \dot{e}_z^U + \\ &+ x \Omega \epsilon_0 \theta_3^{pe} [\alpha_3^{pe} f_3^{pe} (e_z^U) - \xi_3^{pe} p_z^U] |\dot{e}_z^U| \end{aligned} \quad (86)$$

$$\begin{aligned} \dot{m}_y^U &= \Omega (y \partial_x - x \partial_y) m_y^U + \\ &+ y \Omega \epsilon_0 \kappa_3^{pe} |\alpha_3^{pe} f_3^{pe} (e_z^U) - \xi_3^{pe} p_z^U| \dot{e}_z^U + \\ &+ y \Omega \epsilon_0 \theta_3^{pe} [\alpha_3^{pe} f_3^{pe} (e_z^U) - \xi_3^{pe} p_z^U] |\dot{e}_z^U| \end{aligned} \quad (87)$$

$$\dot{m}_z^U = \Omega (y\partial_x - x\partial_y) m_z^U \quad (88)$$

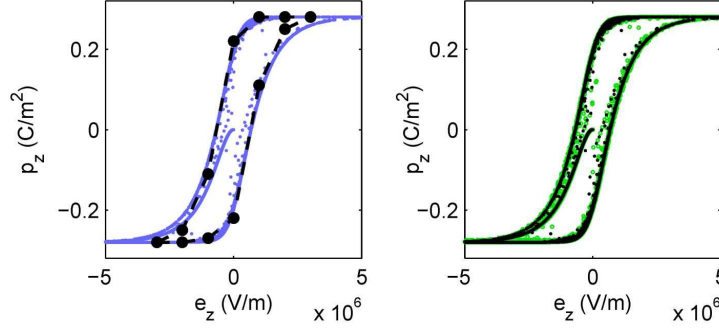
where

$$p_3^V = p_z^U - \frac{\Omega}{c^2} [xm_x^U + ym_y^U] \quad (89)$$

$$e_3^V = e_z^U - \Omega [xb_x^U + yb_y^U] \quad (90)$$

$$\mathcal{E}_3^V = \dot{e}_z^U - \Omega [yb_y^U + xb_x^U] + \Omega (x\partial_y - y\partial_x) e_z^U \quad (91)$$

and terms of order  $(r\Omega/c)^2$  consistently dropped. The time derivative of the magnetic field in (91) is found using the time derivative of (18). From (89) this implies that it is sufficient to replace  $p_3^V$  with  $p_z^U$  in (86) and (87). Similarly (90) implies that it is sufficient to replace  $e_3^V$  with  $e_z^U$  in (85), (86) and (87) and, to order  $(r\Omega/c)^2$ ,  $\Omega p_3^V \equiv \Omega p_z^U$  and  $\Omega e_3^V \equiv \Omega e_z^U$ .



**Figure 2.** For small amplitude drive currents in the cavity hysteretic polarization loci will exhibit different features at different locations in the ferroelectric medium. For sufficient large amplitude drive currents all such loci can be driven to limit cycles that exhibit saturation. In the computations the model parameters do not vary with position in the medium so the resulting limit cycles are independent of position. A typical ferroelectric hysteresis loop on the left displays the evolution of the  $z$ -component of electrical polarization inside the cylinder, at rest at  $z = 1.36\text{m}$ . This history is induced by the  $z$ -component of the time-dependent electric field driven by the external initially zero 250Hz drive current in the cavity. The evolution proceeds from the origin where  $p_z$  and  $e_z$  are zero to a time where a limit cycle is in evidence. The parameters of the model ( $\alpha_z^{pe} = 3.6 \times 10^4$ ,  $\beta_z^{pe} = 2.0 \times 10^{-6}\text{m/V}$ ,  $\xi_z^{pe} = 1.3 \times 10^5\text{m}^2/\text{C}$ ,  $\kappa_z^{pe} = 0.5$ ,  $\theta_z^{pe} = 0.5$ ,  $\sigma = 2.6 \times 10^{-4}\text{S/m}$ ) have been chosen so that experimental data points (dark circles) from reference [9] lie close to this particular limit cycle. The points on the left result when the computation is repeated but with the cylinder rotating at 1497rpm. In this simulation the changes in the  $p_z$  vs  $e_z$  hysteresis loops induced by rotation are imperceptible on the scales employed in the figure.

## 6. Numerical Analysis

The above constitutive equations for components of the induced polarisation in the rotating ferroelectric are coupled to the macroscopic Maxwell equations described in section 2. If we assume that initially  $p_x^U = 0$ ,  $p_y^U = 0$  and  $m_z^U = 0$  it follows from (85), (86) and (87) that the complete differential system becomes

$$\dot{p}_z^U = \Omega (y\partial_x - x\partial_y) p_z^U + \epsilon_0 \{ \alpha_3^{pe} \tanh(\beta_3^{pe} e_3^V) - \xi_3^{pe} p_3^V \} \cdot \{ \kappa_3^{pe} \mathcal{E}_3^V \operatorname{sgn}(\epsilon_0 [\alpha_3^{pe} \tanh(\beta_3^{pe} e_3^V) - \xi_3^{pe} p_3^V]) + \theta_3^{pe} |\mathcal{E}_3^V| \} \quad (92)$$

$$\begin{aligned} \dot{m}_x^U &= \Omega (y\partial_x - x\partial_y) m_x^U + \\ &\quad x\Omega\epsilon_0\kappa_3^{pe} |\alpha_3^{pe} \tanh(\beta_3^{pe} e_z^U) - \xi_3^{pe} p_z^U| \dot{e}_z^U + \\ &\quad x\Omega\epsilon_0\theta_3^{pe} [\alpha_3^{pe} \tanh(\beta_3^{pe} e_z^U) - \xi_3^{pe} p_z^U] |\dot{e}_z^U| \end{aligned} \quad (93)$$

$$\begin{aligned} \dot{m}_y^U &= \Omega (y\partial_x - x\partial_y) m_y^U + \\ &\quad y\Omega\epsilon_0\kappa_3^{pe} |\alpha_3^{pe} \tanh(\beta_3^{pe} e_z^U) - \xi_3^{pe} p_z^U| \dot{e}_z^U + \\ &\quad y\Omega\epsilon_0\theta_3^{pe} [\alpha_3^{pe} \tanh(\beta_3^{pe} e_z^U) - \xi_3^{pe} p_z^U] |\dot{e}_z^U| \end{aligned} \quad (94)$$

and

$$\epsilon_0 \dot{e}_x^U = (\nabla \times \mathbf{H}^U)_x - \sigma e_x^U - j_{ext,x}^U \quad (95)$$

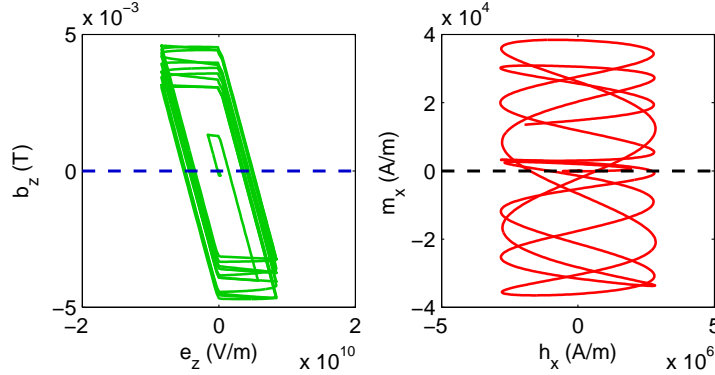
$$\epsilon_0 \dot{e}_y^U = (\nabla \times \mathbf{H}^U)_y - \sigma e_y^U - j_{ext,y}^U \quad (96)$$

$$\epsilon_0 \dot{e}_z^U = (\nabla \times \mathbf{H}^U)_z - \sigma e_z^U - j_{ext,z}^U - \dot{p}_z^U \quad (97)$$

$$\dot{h}_x^U = \frac{-1}{\mu_0} (\nabla \times \mathbf{E}^U)_x + \dot{m}_x^U \quad (98)$$

$$\dot{h}_y^U = \frac{-1}{\mu_0} (\nabla \times \mathbf{E}^U)_y + \dot{m}_y^U \quad (99)$$

$$\dot{h}_z^U = \frac{-1}{\mu_0} (\nabla \times \mathbf{E}^U)_z \quad (100)$$



**Figure 3.** These graphs display histories of computed field components located at the cavity point  $(x, y, z) = (2.45\text{m}, 2.45\text{m}, 1.36\text{m})$  (which is inside the rotating ferroelectric cylinder) driven by a steady external 250Hz current outside the cylinder. During the displayed history the cylinder performs 1 complete revolution. The graph on the left displays the evolution of the  $z$ -components  $b_z$  and  $e_z$ , located at this cavity point, projected onto a region of the  $b_z$ - $e_z$  plane. On the right a non-zero magnetization  $m_x$  induced by the time-dependent  $h_x$  field is displayed. Similar behaviour is computed for  $m_y$  as a function of the magnetic field and indicates the presence of a rotation induced hysteretic magnetisation in the ferroelectric medium that increases in magnitude with the rotation speed  $\Omega$ .

To analyze these equations numerically Yee's algorithm [10],[11] for standard materials on a standard staggered  $E-H$  mesh has been extended to include non-linear hysteretic media with branched solutions. Since  $\nabla \cdot \mathbf{B}^U$  and  $\nabla \cdot \mathbf{D}^U$  in the cavity are chosen initially zero and the external current is constructed to be divergenceless the full set of material Maxwell equations are accommodated. The results of this analysis are summarized in Figures 1, 2 and 3.

## 7. Conclusions

A class of non-linear constitutive relations for materials with memory has been discussed in the framework of covariant macroscopic Maxwell theory. The general approach enables models to be formulated for arbitrarily moving media including those that exhibit hysteretic responses to time varying electromagnetic fields. Using a particular parameterized model, consistent with experimental data for a particular material that exhibits purely ferroelectric hysteresis when at rest in a slowly varying electric field, a numerical analysis of its response to a driven harmonic electromagnetic field in a rectangular cavity has been performed when in different states of rotation. The results indicate that such a model offers a means to compute numerically the significance of induced hysteretic magnetisation in a ferroelectric medium as a function of its rotation speed and the frequency of a self-consistent electromagnetic field.

## 8. Acknowledgements

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